Lattice-independent approach to thermal phase mixing

Carmen J. Gagne* and Marcelo Gleiser†
Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755
(Received 13 October 1999)

We show how to achieve lattice-spacing-independent results in numerical simulations of finite-temperature stochastic scalar field theories. We generalize a previous approach by obtaining results which are independent of the renormalization scale. As an application of our method, we examine thermal phase mixing in the context of Ginzburg-Landau models with short-range interactions. In particular, we obtain the lattice-spacing and renormalization-scale-independent critical value of the control parameter which determines the free-energy barrier between the two low-temperature phases. We also propose a simple procedure to extract the critical value of control parameters for different choices of lattice spacing.

PACS number(s): 05.10.−a, 05.70.Jk, 11.10.Wx

I. INTRODUCTION

During the past decade, the study of equilibrium and nonequilibrium dynamics of field theories has greatly benefited from the widespread availability of workstations capable of millions of floating point operations per CPU second.

One of the most popular applications of computers in the physical sciences is the examination of phenomena that are generated by nonperturbative effects. These include nonlinear dynamical systems with a few, several, or an infinite number of degrees of freedom. Of these, we are particularly interested here in the latter, as they represent a unique challenge to computational physics. Implementing field theories in the computer implies discretizing not only time but also space: the system is cast on a finite lattice with a discrete spatial step, effectively cutting off the theory both in the infrared (by the lattice size) and in the ultraviolet (by the lattice spacing). Although in classical field theories an ultraviolet cutoff solves the Rayleigh-Jeans ultraviolet catastrophe, the solution comes with a high price tag: whenever there is dynamical mixing of short and long wavelength modes, the results will in general depend on the shortest distance scale in the simulation, the lattice spacing. To be sure, in many instances this dependence on small spatial scales does not affect qualitatively the physics one is interested in: for example, very near criticality for Ising systems, where spatial correlations in the order parameter diverge [1], or are controllable in some way [2]. However, in many other cases one is interested in achieving a proper continuum limit on the lattice that is independent of the choice of ultraviolet cutoff. These include a wide range of phenomena that have triggered much recent interest, from pattern formation in fluid dynamics [3] to simulations of phase transitions and topological defect formation, which often use stochastic methods [4].

In the present paper, we are concerned with curing, or at least greatly alleviating, the lattice-spacing dependence that appears in stochastic simulations of scalar field theories. We will show that it is indeed possible to obtain results that are lattice-spacing independent, as long as proper counterterms are added to the lattice effective potential. Following a suggestion by Parisi [5], lattice-spacing-independent results were recently obtained by Borrill and one of us within the context of finite-temperature symmetry restoration in a simple Ginzburg-Landau model [6]. However, that study focused on a regime where the large temperatures needed for symmetry restoration compromised the approach to obtain lattice-spacing-independent results, which is based on a perturbative expansion in powers of the temperature. Furthermore, no attempt was made to obtain results that were independent of the renormalization scale. Thus, in that study, the numerical prediction for the critical temperature depends on the particular choice of renormalization scale.

Here, we would like to apply an expanded version of the method proposed in Ref. [6] to a related problem, phase mixing in Ginzburg-Landau models. The distinction between phase mixing and symmetry restoration is made clearer through the following argument. Suppose a system described by a Ginzburg-Landau free energy density with *odd* powers of the order parameter $\phi(t,\mathbf{x})$ is rapidly cooled from a high temperature to a temperature where two phases can coexist. The system was cooled so as to remain entirely in one of the two phases. The odd term(s) could be due to an external magnetic field (linear term), or to the integration of other fields coupled to ϕ, as in certain gauge theories (cubic term) [7], or in de Gennes–Landau models of the nematic-isotropic transition in liquid crystals [8]. Due to the odd terms, there is a free-energy barrier for large-amplitude fluctuations between the two phases. [Of course, small-amplitude fluctuations within each phase are also possible, but less interesting.] This barrier is usually controlled by the coefficients of the odd terms in the Ginzburg-Landau model, which we will call the control parameter(s).

Suppose now that the system is held at the temperature where the two phases have the same free energy densities (sometimes called the critical temperature in the context of discontinuous phase transitions) and that we are free to change the value of the control parameter(s). The question we would like to address is how does the system behave as a function of the control parameter(s), that is, as the free-energy barrier for large-amplitude fluctuations between the two phases is varied. As is well known, the mean-field

*Electronic address: carmen.gagne@dartmouth.edu
†Electronic address: marcelo.gleiser@dartmouth.edu

theory approach breaks down when large-amplitude fluctuations about equilibrium become large enough. Thus, we should expect that the prediction from the Ginzburg-Landau model, that the system remains localized in one phase until the barrier disappears (when the control parameter goes to zero) will eventually be wrong. There will be a critical value for the control parameter beyond which nonperturbative effects lead to the mixing of the two phases. (Note that due to the odd terms, there is no symmetry to be restored.) In the language of the Ginzburg-Landau model, the system should at this point be described as having a single well, centered at the mean value of the order parameter. We would like to obtain the lattice-independent critical value of the control parameter for this phase mixing to occur.

It is important to distinguish between phase coexistence and phase mixing. As is well-known, phase coexistence will generally occur when a system is cooled into the so-called phase coexistence region of the phase diagram. In this case, the system will relax into its lowest free-energy configuration via spinodal decomposition. Here, we are preparing the system initially outside the phase coexistence region, namely in one particular phase only. In the infinite-volume limit, mean field theory predicts the system will remain there, since, as the two minima are degenerate, the nucleation of a critical droplet would cost an infinite amount of free energy. Phase mixing is a nonperturbative phenomenon characterized by large-amplitude fluctuations not included in the mean-field approach. It signals the breakdown of mean-field theory.

This paper is organized as follows: In the next section, we describe the continuum model we use and some of its properties. In Sec. III we describe the lattice implementation and how simulations using a bare lattice potential give results that depend severely on the lattice spacing. In Sec. IV we show how to cure this dependence, and also how to make the simulations independent of the choice of renormalization scale. We conclude in Sec. V with a brief summary of our results and a discussion of future work.

II. THE MODEL IN THE CONTINUUM

Our starting point is the two-dimensional Hamiltonian, (we use $c=k_B=1$)

$$\frac{H[\phi]}{T}=\frac{1}{T}\int d^2x\left[\frac{1}{2}(\nabla\phi\cdot\nabla\phi)+V(\phi)\right], \tag{1}$$

where the homogeneous part of the free energy density is

$$V(\phi)=\frac{a}{2}(T^2-T_2^2)\phi^2-\frac{\alpha}{3}T\phi^3+\frac{\lambda}{4}\phi^4. \tag{2}$$

This choice of $V(\phi)$ is inspired by several models of nucleation in the condensed matter [9] and high-energy physics literature, in particular in recent models of the electroweak phase transition [7]. The several parameters in $V(\phi)$ allow one to apply it to several situations of interest. However, we note that here the order parameter is a scalar quantity, and thus the critical behavior of this model belongs to the universality class of the two-dimensional Ising model [10,6]. It is quite straight forward to generalize our results to systems in different numbers of spatial dimensions.

At the critical temperature $T_c^2=T_2^2/(1-2\alpha^2/9a\lambda)$, the system exhibits two degenerate free energy minima at

$$\phi=0 \quad \text{and} \quad \phi_+=\frac{2\alpha T_c}{3\lambda}, \tag{3}$$

while at the temperature T_2 the barrier between the two phases disappears. Throughout this paper, we will be interested in the behavior of the system at T_c. One reason for this choice has to do with the use of a perturbative expansion, which is in powers of T; at T_c the expansion parameter is sufficiently small, allowing us to stay at 1 loop. Another reason is that we are interested in measuring the breakdown of mean-field theory in terms of parameters controlling the free-energy barrier, and the calculations are much simpler at T_c, as we will see next.

According to the model described by Eq. (2), at T_c, unless $\alpha=0$ (or $\lambda\to\infty$) there will always be a barrier separating the two phases: at T_c, this model does not predict phase mixing to occur. It is thus very convenient to introduce the shifted field

$$\phi\to\phi'\equiv\phi-\frac{T\alpha}{3\lambda}, \tag{4}$$

and write the shifted homogeneous free energy density as (dropping the primes)

$$V_0(\phi)=-\frac{1}{2}\mu^2(T)\phi^2+\frac{\lambda}{4}\phi^4+A(T)\phi+\text{constants}, \tag{5}$$

with

$$\mu^2(T)\equiv-a(T^2-T_2^2)+\frac{T^2\alpha^2}{3\lambda}, \tag{6}$$

and

$$A(T)\equiv a(T^2-T_2^2)\frac{T\alpha}{3\lambda}+\frac{2}{27}\frac{T^3\alpha^3}{\lambda^2}. \tag{7}$$

The shifted free energy density is just the usual Ginzburg-Landau free energy density with an external magnetic field $A(T)$. Note that $A(T_c)=0$ and the two minima are degenerate, as they should be. We now introduce the dimensionless variables $\theta\equiv T/T_2$, $\tilde{t}\equiv\sqrt{a}T_2t$, $\tilde{x}\equiv\sqrt{a}T_2x$, $\tilde{\phi}\equiv\phi/\sqrt{T_2}$, according to which we can write, at $\theta_c=[1-2\tilde{\alpha}^2/9\tilde{\lambda}]^{-1/2}$,

$$\tilde{V}_0=-\frac{1}{2}\tilde{\mu}^2(\theta_c)\tilde{\phi}^2+\frac{1}{4}\tilde{\lambda}\tilde{\phi}^4, \tag{8}$$

where $\tilde{\lambda}\equiv\lambda/aT_2$, $\tilde{\alpha}\equiv\alpha/a\sqrt{T_2}$, and

$$\tilde{\mu}^2=\frac{\mu^2}{aT_2}=-(\theta_c^2-1)+\frac{\theta_c^2\tilde{\alpha}^2}{3\tilde{\lambda}}. \tag{9}$$

Since we will keep the system at $\theta_c(\tilde{\alpha},\tilde{\lambda})$, the only two control parameters are $\tilde{\alpha}$ and $\tilde{\lambda}$. In what follows, we will fix $\tilde{\lambda}=0.1$ for simplicity. This was also the choice in a previous study of phase mixing in the same system, which did not

address the issue of lattice-spacing dependence [11]. We will also drop all tildes, except in the plots and captions, where unshifted, dimensionless variables are used and marked explicitly.

III. NUMERICAL RESULTS: BARE LATTICE

A. Description of the simulation

As mentioned in the Introduction, we would like to study the behavior of the system described in the previous section when coupling to an external thermal bath promotes fluctuations about equilibrium. We will consider the situation where the system is initially prepared in the phase given by $\phi=0$ in the unshifted potential or, more generically, the left well. Since we are only interested in the final equilibrium value of the system, we will simulate the coupling of the scalar field ϕ to the thermal bath using a generalized Langevin equation,

$$\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi-\eta\frac{\partial\phi}{\partial t}-\frac{\partial V_0}{\partial\phi}+\xi(\mathbf{x},t), \tag{10}$$

where the viscosity coefficient η, set equal to unity in all simulations, is related to the stochastic force of zero mean $\xi(\mathbf{x},t)$ by the fluctuation-dissipation relation,

$$\langle\xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle=2\eta\theta\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{11}$$

The system is discretized and put on a square lattice with side length, L, equal to 64 for all the simulations, but several lattice spacings, δx, and time steps, δt, are used. For $\delta x=1.0$, 0.8, and 0.2 the respective time steps are $\delta t=0.2$, 0.1, and 0.02. We have, of course, checked the stability of the program for these choices of lattice parameters. Using a standard second-order staggered leapfrog method (which is second order in both space and time) we can write,

$$\dot\phi_{i,m+1/2}=\frac{\left(1-\frac{1}{2}\eta\delta t\right)\dot\phi_{i,m-1/2}+\delta t(\nabla^2\phi_{i,m}-V_0'(\phi_{i,m})+\xi_{i,m})}{1+\frac{1}{2}\eta\delta t}$$

$$\phi_{i,m+1}=\phi_{i,m}+\delta t\dot\phi_{i,m+1/2} \tag{12}$$

where i indices are spatial and m indices temporal, overdots represent derivatives with respect to t and primes with respect to ϕ. The discretized fluctuation-dissipation relation now reads

$$\langle\xi_{i,m}\xi_{j,n}\rangle=2\eta\theta\frac{\delta_{i,j}}{\delta x^2}\frac{\delta_{m,n}}{\delta t}, \tag{13}$$

so that

$$\xi_{i,m}=\sqrt{\frac{2\eta\theta}{\delta x^2\delta t}}G_{i,m}, \tag{14}$$

where $G_{i,m}$ is taken from a zero-mean unit-variance Gaussian.

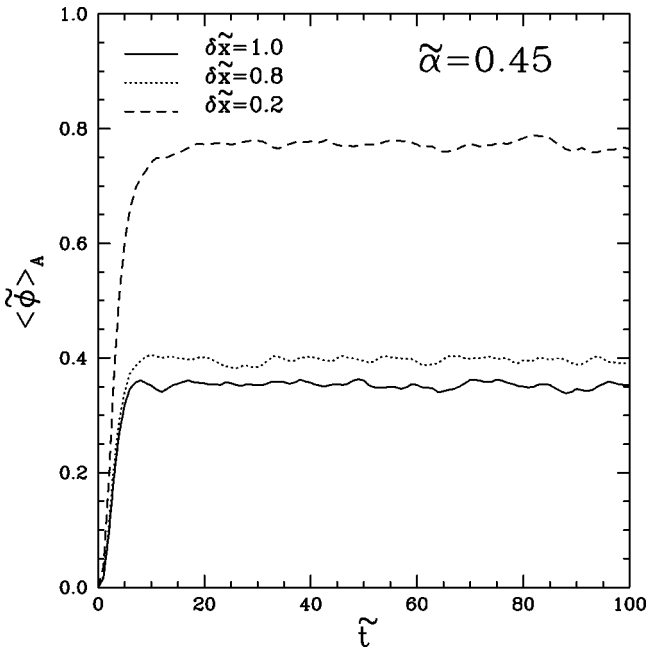

FIG. 1. $\langle\tilde\phi\rangle_A(\tilde t)$ for the bare potential, for several choices of lattice spacing, for $\tilde\alpha=0.45$.

B. Results from bare lattice simulations

Keeping the system always at the critical temperature θ_c, we are interested in its behavior as the free-energy barrier between the two equilibrium phases is changed. We will measure the value of the ensemble-averaged and area-averaged order parameter $\langle\phi\rangle_A(t)\equiv 1/A\int d^2x\,\phi(\mathbf{x},t)$ for several choices of the lattice spacing δx, taking note of its final equilibrium value, $\overline{\phi}_{\rm eq}$. In Fig. 1 we show the results for $\langle\phi\rangle_A(t)$ for several choices of lattice spacing and $\alpha=0.45$. The dependence on lattice spacing is quite evident; different lattices produce different physics.

In Fig. 2, we show the phase diagram depicting phase mixing as a function of α for different choices of the lattice spacing δx. The phase diagram is constructed by defining the ''phase-mixing order parameter,''

$$\delta_\phi(\alpha)\equiv|\overline{\phi}_{\rm eq}-\phi_{\rm max}|/\phi_{\rm max}, \tag{15}$$

where $\phi_{\rm max}=\alpha\theta_c/3\lambda$ is the location of the maximum of the free energy density separating the two phases. Clearly, as α decreases, the free-energy barrier decreases and larger-amplitude fluctuations between the two phases become more probable. Below a critical value α_c, $\overline{\phi}_{\rm eq}$ just tracks the location of the maximum, indicating complete phase mixing, or the breakdown of the mean field theory of Eq. (1).

The problem, though, is that phase mixing, or the breakdown of mean-field theory, occurs for values of α_c, which are strongly dependent on the value of δx, as can be seen from Fig. 2. For the range of δx investigated, $0.2\leqslant\delta x\leqslant 1$, we obtained $0.355\lesssim\alpha_c\lesssim 0.40$. In the next section, we argue that this dependence can be effectively cured by including proper counterterms to the lattice potential.

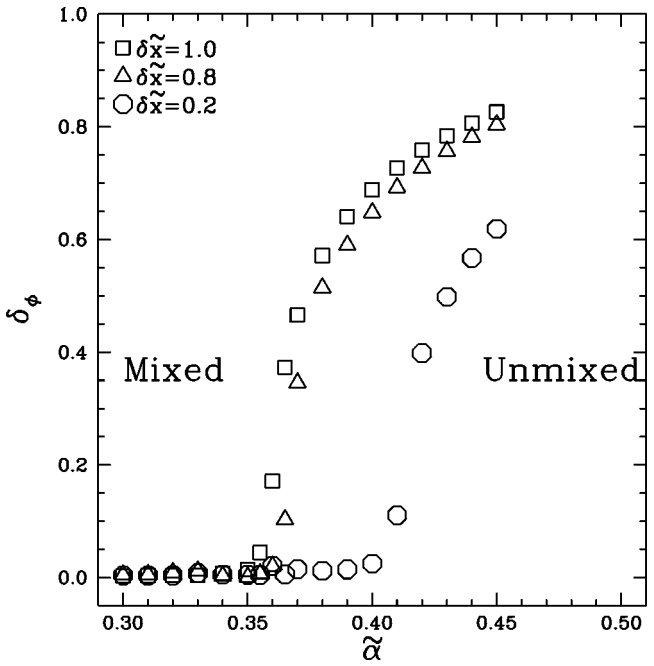

FIG. 2. Phase diagram for the bare potential for various lattice spacings.

IV. APPROACHING THE CONTINUUM ON THE LATTICE

A. Computing the lattice effective potential

Setting up a continuum system on a lattice introduces two artificial length scales, the ultraviolet momentum cutoff $\Lambda = \pi/\delta x$ and the infrared momentum cutoff $\Lambda_L = \pi/L$, where L is the lattice size. In the continuum limit, $L \to \infty$, and $\delta x \to 0$ or, equivalently, the number of degrees of freedom $N = (L/\delta x)^d \to \infty$. The coupling to the thermal bath induces fluctuations at all allowed length scales. We should thus expect that the lattice simulation is related to a continuum model with both infrared and ultraviolet cutoffs. In order to obtain the lattice effective potential, we start by analyzing the divergences of the related continuous model.

For classical field theories in two dimensions, the corresponding 1-loop corrected effective potential is given by

$$V_{1\,\mathrm{L}}(\phi) = V_0(\phi) + \frac{T}{2}\int_{\Lambda_L}^{\Lambda} \frac{d^2p}{(2\pi)^2} \ln(p^2 + V_0'') + \text{counterterms}, \tag{16}$$

where the primes denote derivatives with respect to ϕ. Performing the integration and making all variables dimensionless we obtain,

$$V_{1\,\mathrm{L}}(\phi) = V_0(\phi) + \frac{\theta}{8\pi} V_0'' \left[1 - \ln\left(\frac{\Lambda_L^2 + V_0''}{\Lambda^2} \right) \right] - \frac{\theta}{8\pi} \Lambda_L^2 \ln(\Lambda_L^2 + V_0'') + B\phi^2 + \text{constants}. \tag{17}$$

The infrared cutoff does not introduce a divergence as $\Lambda_L \to 0$, but it does introduce finite corrections to $V_{1\,\mathrm{L}}$, or finite size effects, which become small as L increases. These become more severe near criticality, but well-known scaling behavior can be used to regulate this dependence [12]. As we will further argue below, for our purposes we can safely set $\Lambda_L = 0$. This is not the case for the ultraviolet cutoff. The reader can see now why it is useful to use the shifted potential of Eq. (5) as opposed to the original one of Eq. (2): all divergences are quadratic in ϕ, simplifying the computations considerably, while the physical results, of course, remain unchanged. This is why we added only the counterterm $B\phi^2$ above.

The counterterm B is computed by imposing the renormalization condition

$$V_{1L}''(\phi_{\mathrm{RN}}) = V_0''(\phi_{\mathrm{RN}}) = M^2, \tag{18}$$

where M is the arbitrary renormalization scale and we write $\phi_{\mathrm{RN}} \equiv \sqrt{(M^2+\mu^2)/3\lambda}$. [Note that M here is dimensionless (tilde is dropped), being defined as $\tilde{M} = M/T_2$.] One obtains,

$$B(M) = \frac{\theta}{16\pi} \left[V_0'''' \ln\left(\frac{V_0''}{\Lambda^2} \right) + \frac{(V_0''')^2}{V_0''} \right]_{\phi=\phi_{\mathrm{RN}}}. \tag{19}$$

Applying this to the shifted potential of Eq. (5), we obtain, for the 1-loop renormalized continuum potential,

$$V_{1L}^M(\phi) = \left[-\frac{1}{2}\mu^2 + \frac{9\lambda\theta}{8\pi} + \frac{3\lambda\theta\mu^2}{4\pi M^2} \right] \phi^2 + \frac{\lambda}{4}\phi^4 + A\phi - \frac{3\lambda\theta}{8\pi}\phi^2 \ln\left(\frac{-\mu^2 + 3\lambda\phi^2}{M^2} \right) + \frac{\mu^2\theta}{8\pi} \ln(-\mu^2 + 3\lambda\phi^2) + \text{constants}. \tag{20}$$

Recall that at θ_c the linear term proportional to $A(\theta)$ vanishes. Since the counterterm cancels the dependence on the ultraviolet cutoff, we define the lattice effective potential as [6]

$$V_{\mathrm{latt}}(\phi) = V_0(\phi) + B(M)\phi^2. \tag{21}$$

In Fig. 3 we show the results of repeating the simulations of Fig. 1 but now adding the counterterm to the lattice simulations following Eq. (21). The addition of the counterterm practically eliminates the lattice-spacing dependence of the results. Figure 3 also shows the near elimination of lattice-spacing dependence for $\alpha = 0.40$.

B. Extracting the critical value of the order parameter

In Figs. 4 and 5, we show the phase diagrams using δ_ϕ defined in Eq. (15) as a function of α for different choices of lattice spacing δx. These are to be compared with Fig. 2. Figure 4 is for a choice of renormalization scale $M = 1$, while Fig. 5 is for $M = 10$. It is clear that the results for different lattice spacings converge around one value of α_c.

We compute α_c as follows: for a given value of α we perform several ($i_{\max}$) measurements of $\overline{\phi}_{\mathrm{eq}}$ by varying the lattice spacing, which we call $\overline{\phi}_{\mathrm{eq}}^{i}(\alpha)$. Their average is simply $\langle \overline{\phi}_{\mathrm{eq}}(\alpha) \rangle = [\Sigma_1^{i_{\max}} \overline{\phi}_{\mathrm{eq}}^{i}(\alpha)]/i_{\max}$, while the departure from

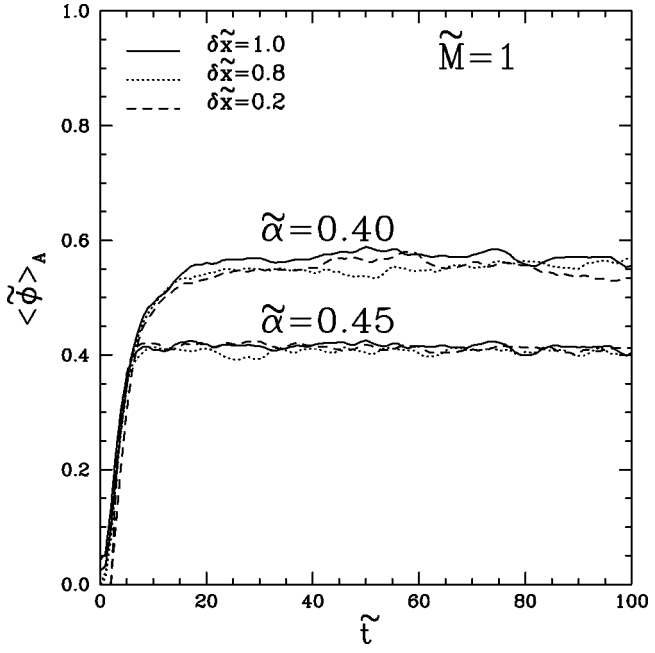

FIG. 3. $\langle\tilde{\phi}\rangle_A(\tilde{t})$ with the counterterm added and $\tilde{M}=1$, for several choices of lattice spacing, for $\tilde{\alpha}=0.45$ and $\tilde{\alpha}=0.40$.

the average for each measurement is, $\Delta\overline{\phi}^i_{\rm eq}=|\overline{\phi}^i_{\rm eq}-\langle\overline{\phi}_{\rm eq}\rangle|/\langle\overline{\phi}_{\rm eq}\rangle$.

Near criticality, the results are naturally poorer due to the existence of long-range correlations in the field. We can use this fact to our advantage, since we expect that, at criticality, the departure from the average defined above is maximized, that is, the quantity

$$\langle\Delta\phi_{\rm eq}(\alpha)\rangle\equiv\frac{\sum_1^{i_{\rm max}}\Delta\overline{\phi}^i_{\rm eq}}{i_{\rm max}}, \tag{22}$$

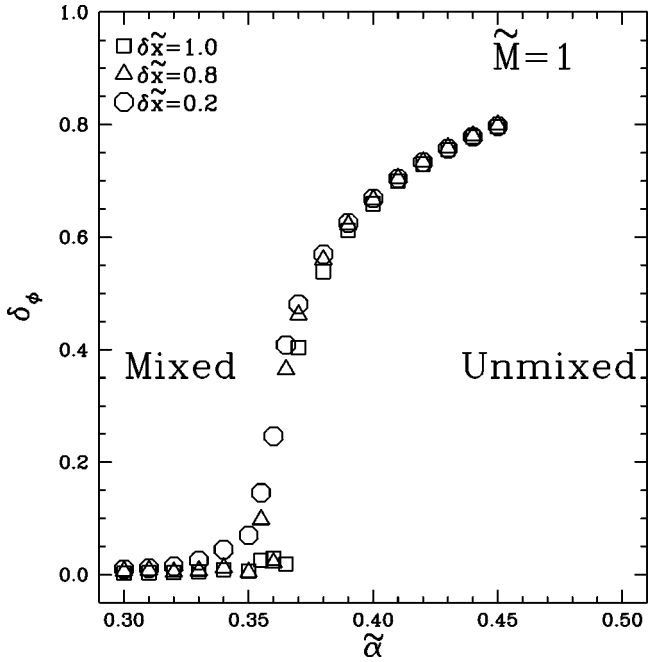

FIG. 4. Phase diagram for $\tilde{M}=1$.

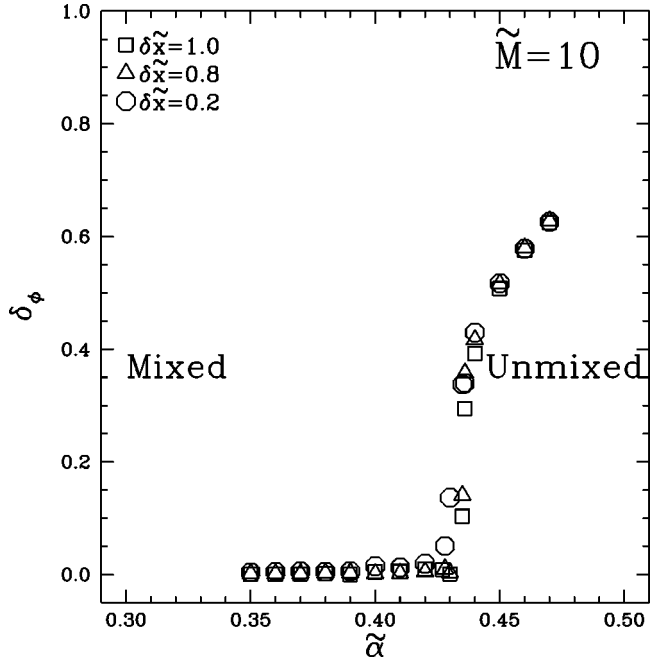

FIG. 5. Phase diagram for $\tilde{M}=10$.

reaches a maximum at α_c. This can be clearly seen from Fig. 6 for the same choices of lattice spacings (or coarse-graining scales) as in Figs. 4 and 5. The measured value of α_c is now $\alpha_c\simeq 0.365\pm 0.005$, for $M=1$, and $\alpha_c\simeq 0.435\pm 0.005$ for $M=10$.

We have thus achieved lattice-spacing independence on the measurement of α_c. Clearly, the error in α_c could be further decreased by taking a larger number of measurements of $\overline{\phi}^i_{\rm eq}$. However, since our main goal here is to show the convergence of the results for different lattice spacings, we are not concerned with very high-accuracy measurements. Nevertheless, the values for α_c still depend on the renormalization scale, which is arbitrary. In the next subsection, we show how to obtain lattice results that are independent of M.

C. Achieving independence of renormalization scale on the lattice

As with conventional renormalization theory, the renormalized potential should not depend on the choice of renormalization scale [13]. One usually solves the renormalization group equations to find how the couplings vary with the scale. Here, we propose a simpler approach that works quite well on the lattice implementation of scalar field theories. It is an interesting question how to generalize it to more complex models.

Consider the 1-loop renormalized potential $V^M_{1\,\rm L}(\phi)$ as given in Eq. (20). The superscript M is a reminder that this potential is renormalized at a given scale M. Now consider an equivalent potential renormalized at another scale M', $V^{M'}_{1\,\rm L}(\phi)$. Since the divergences are quadratic, this potential has a shifted mass μ'^2. By imposing that the two potentials are identical, $V^M_{1\,\rm L}(\phi)=V^{M'}_{1\,\rm L}(\phi)$, we obtain a condition on the shifted mass μ'^2, approximating $\ln(-\mu'^2+3\lambda M')\simeq\ln(-\mu^2+3\lambda M)$,

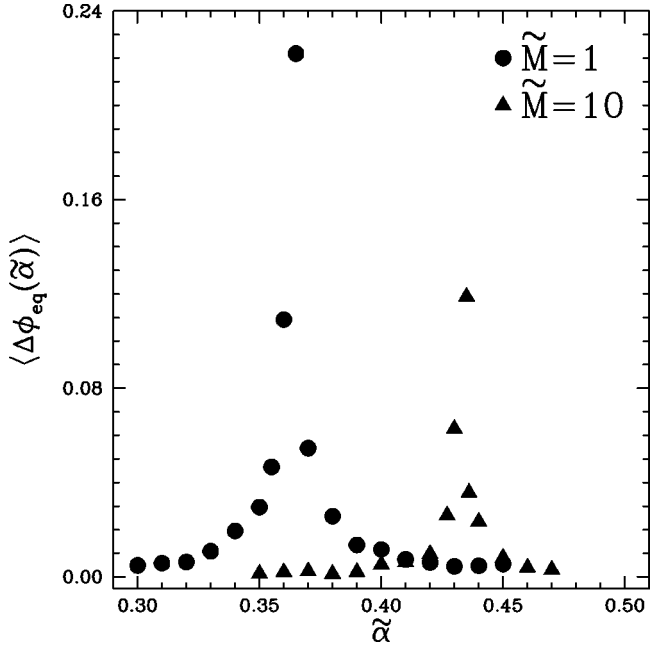

FIG. 6. $\langle \Delta\phi_{\rm eq}(\tilde{\alpha})\rangle$ for $\tilde{M}=1$ and $\tilde{M}=10$, with the respective $\tilde{\alpha}_c$'s at the maxima.

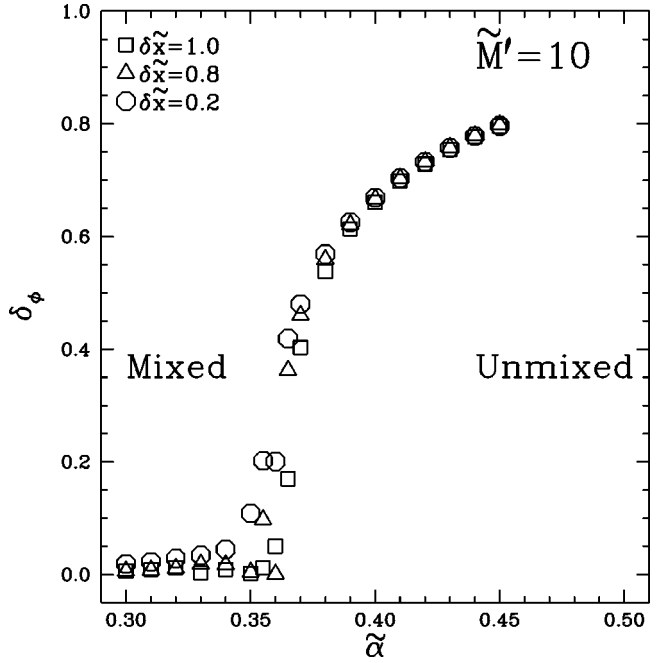

FIG. 7. Phase diagram for $\tilde{M}'=10$.

$$\mu'^2 \simeq \mu^2 + \frac{3\lambda\theta}{4\pi}\ln\left(\frac{M'^2}{M^2}\right) - \frac{3\lambda\theta\mu^2}{2\pi}\left[\frac{1}{M^2}-\frac{1}{M'^2}\right]. \tag{23}$$

Thus, we can always relate a theory with a choice of M to any other theory with M' by redefining the mass μ^2 according to Eq. (23). We claim that this is also the case for the lattice effective potential.

As an illustration, we show the phase diagram for $M'=10$ in Fig. 7, where the results for $M'=10$ were obtained after scaling μ^2 according to Eq. (23) in the lattice potential of Eq. (21). It is practically indistinguishable from the phase diagram for $M=1$ shown in Fig. 4. Figure 8 demonstrates clearly that $M'=10$ has the identical α_c previously found for $M=1$, within our level of accuracy. This is in stark contrast to Fig. 6, where the values of α_c for $M=1$ and $M=10$ were very different, as evidenced by its ''twin peaks'' structure.

V. SUMMARY AND OUTLOOK

We have investigated the continuum limit of lattice simulations of stochastic scalar field theories. In particular, we have proposed a method to obtain not only lattice-spacing independent results, but also results independent of the renormalization scale of the lattice effective potential. We illustrated our approach by examining a Ginzburg-Landau model which exhibits phase mixing depending on the values of the parameters controlling the free-energy barrier for large-amplitude fluctuations between the two low-temperature phases in our model. Thermal fluctuations of the order parameter are induced by coupling it to a thermal bath at fixed temperature T_c, defined as the temperature where the two phases have the same free energy density. We simulate the dynamics using a generalized Langevin equation with Gaussian noise, which brings the system to its final equilibrium state.

The results were presented in terms of phase diagrams which clearly illustrate the effectiveness of our approach. We also proposed a simple way of determining the critical value of the control parameter for phase mixing, which uses the spread in values of the equilibrium order parameter around criticality for different choices of lattice spacing (or coarse-graining scales). Thus, we effectively turn a weakness of lattice simulations into a strength, something that can be useful for the examination of critical phenomena of continuous field theories in fairly small lattices.

We plan to expand the present study to investigate the effects of spatio-temporal memory on the dynamics of non-

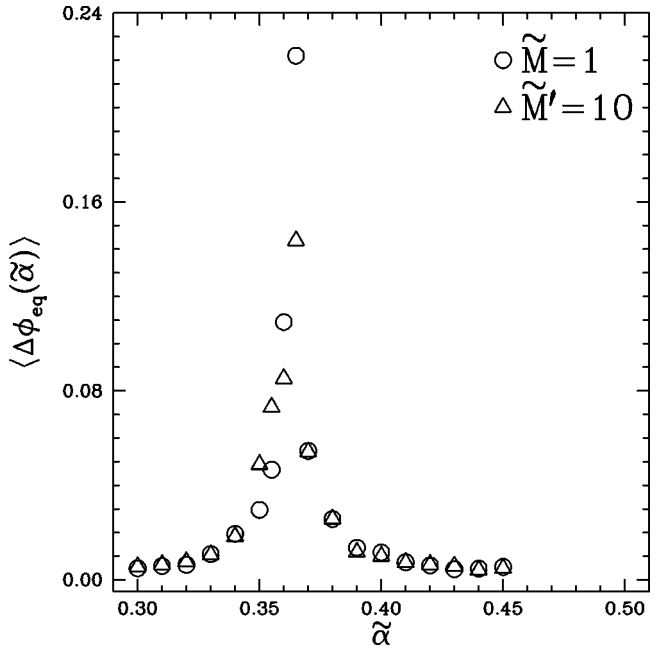

FIG. 8. $\langle \Delta\phi_{\rm eq}(\tilde{\alpha})\rangle$ for $\tilde{M}=1$ and $\tilde{M}'=10$ showing the same value of $\tilde{\alpha}_c$.

equilibrium fields. Recent results have shown that the effective Langevin equation for self-coupled scalar systems exhibits colored and multiplicative noise [14]. It is possible to expand the two-point function characterizing the noise (or noises) in terms of a ''persistence factor,'' which defines short or long-term memory, spatial, temporal, or both. The possible impact of this kind of noise on the nonequilibrium dynamics of fields remains largely unexplored.

ACKNOWLEDGMENTS

C.G. was supported in part by NASA through the New Hampshire NASA Space Grant program (Grant No. NGT5-40010). M.G. was supported in part by The NSF through Grant No. PHY-9453431. M.G. thanks the Particle Theory Group at Boston University, where parts of this work were developed, for their hospitality.

[1] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Frontiers in Physics (Addison-Wesley, New York, 1992), Vol. 85.

[2] C. K. Birdsall and A. B. Langdon, *Plasma Physics via Computer Simulation* (McGraw-Hill, New York, 1985).

[3] *External Noise and Its Interaction with Spatial Degrees of Freedom in Nonlinear Dissipative Systems*, edited by C. R. Doering, H. R. Brand, and R. E. Ecke, *Workshop Proceedings* [J. Stat. Phys. **54**, 1111 (1989)]; J. García-Ojalvo, A. Hernández-Machado, and J.M. Sancho, Phys. Rev. Lett. **71**, 1542 (1993); A. Becker and L. Kramer, *ibid.* **73**, 955 (1994), and references therein.

[4] See, e.g., A. Achucarro, J. Borrill, and A. Liddle, Phys. Rev. Lett. **82**, 3742 (1999).

[5] G. Parisi, *Statistical Field Theory* (Addison-Wesley, New York, 1988).

[6] J. Borrill and M. Gleiser, Nucl. Phys. B **483**, 416 (1997).

[7] For a review see M. Quirós, e-print, hep-ph/9901312.

[8] P. G. de Gennes, *The Physics of Liquid Crystals* (Oxford University Press, New York, 1993).

[9] See Refs. [1,8], and [10].

[10] J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983), Vol. 8.

[11] M. Gleiser, Phys. Rev. Lett. **73**, 3495 (1994).

[12] A. Milchev, D.W. Heermann, and K. Binder, J. Stat. Phys. **44**, 749 (1986).

[13] P. Ramond, *Field Theory: A Modern Primer*, 2nd ed. (Addison-Wesley, New York, 1990).

[14] M. Gleiser and R. Ramos, Phys. Rev. D **50**, 2441 (1994); B. L. Hu, J. P. Paz, and Y. Zhang, in *The Origin of Structure in the Universe*, edited by E. Gunzig and P. Nardone (Kluwer Academic, Dordrecht, 1993); D. Lee and D. Boyanovsky, Nucl. Phys. B **406**, 631 (1993); S. Habib, Ann. (N.Y.) Acad. Sci. **706**, 111 (1993).